

ON THE ACCEPTABLE ELEMENTS

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ABSTRACT. In this paper, we study the set $B(G, \mu)$ of acceptable elements for any p -adic group G . We show that $B(G, \mu)$ contains a unique maximal element and is represented by an element in the admissible subset of the associated Iwahori-Weyl group.

INTRODUCTION

Let F be a finite field extension of \mathbb{Q}_p and L be the completion of the maximal unramified extension of F . Let G be a connected reductive algebraic group over F and σ be the Frobenius morphism. We denote by $B(G)$ the set of σ -conjugacy classes of $G(L)$. The set $B(G)$ is classified by Kottwitz in [Ko1] and [Ko2]. This classification generalize the Dieudonné-Manin classification of isocrystals by their Newton polygons.

Let \tilde{W} be the Iwahori-Weyl group of G . Let $\{\mu\}$ be a conjugacy class of characters of G defined over L . Let $\text{Adm}(\mu) \subset \tilde{W}$ be the admissible subset of \tilde{W} ([KR1]) and $B(G, \mu)$ be the finite subset of $B(G)$ defined by the group-theoretic version of Mazur's theorem [Ko2, §6].

The main result of this paper is as follows.

Theorem 0.1. *The set $B(G, \mu)$ contains a unique maximal element and this element is represented by an element in $\text{Adm}(\mu)$.*

For quasi-split groups, this is obvious as the unique maximal element of $B(G, \mu)$ is represented by t^μ . However, it is much more complicated for non quasi-split groups.

This result is an important ingredient in the proof [He3] of Kottwitz-Rapoport conjecture [KR2, Conjecture 3.1] and [Ra, Conjecture 5.2] on the union of affine Deligne-Lusztig varieties. The knowledge of the explicit description of the maximal element of $B(G, \mu)$ is also useful in the study of μ -ordinary locus of Shimura varieties.

Key words and phrases. Newton polygons, p -adic groups, affine Weyl groups.

1. PRELIMINARIES

1.1. We first recall the classification of $B(G)$ obtained by Kottwitz in [Ko1] and [Ko2].

For any $b \in G(L)$, we denote by $[b]$ the σ -conjugacy class of $G(L)$ that contains b . Let $\Gamma_F = \text{Gal}(\bar{L}/F)$ be the absolute Galois group of F . Let $\kappa_G : B(G) \rightarrow \pi_1(G)_{\Gamma_F}$ be the Kottwitz map [Ko2, §7]. This gives one invariant.

Another invariant is given by the Newton map.

Let S be a maximal L -split torus that is defined over F and let T be its centralizer. Since G is quasi-split over L , T is a maximal torus. We also fix a σ -invariant alcove \mathbf{a} in the apartment of G_L corresponding to S .

To an element $b \in G(L)$, we associate its Newton point ν_b . It is a σ -invariant element in the closed dominant chamber $X_*(T)_{\mathbb{Q}}^+$.

By [Ko2, §4.13], the map

$$B(G) \rightarrow X_*(T)_{\mathbb{Q}}^+ \times \pi_1(G)_{\Gamma_F}, \quad b \mapsto (\nu_b, \kappa_G(b))$$

is injective.

The partial order on $B(G)$ is defined as follows. Let $b, b' \in G(L)$, then $[b] \leq [b']$ if $\kappa_G(b) = \kappa_G(b')$ and $\nu_b \leq \nu_{b'}$, i.e., $\nu_{b'} - \nu_b$ is a non-negative \mathbb{Q} -linear combination of positive relative coroots.

1.2. We follow [HR]. Let N be the normalizer of T . The *finite Weyl group* associated to S is $W_0 = N(L)/T(L)$. The Iwahori-Weyl group associated to S is $\tilde{W} = N(L)/T(L)_1$, where $T(L)_1$ denotes the unique Iwahori subgroup of $T(L)$. The Frobenius morphism σ induces actions on W and \tilde{W} , which we still denote by σ .

Let $\Gamma = \text{Gal}(\bar{L}/L)$. The Iwahori-Weyl group \tilde{W} contains the affine Weyl group W_a as a normal subgroup and

$$\tilde{W} = W_a \rtimes \Omega,$$

where $\Omega \cong \pi_1(G)_{\Gamma}$ is the normalizer of the alcove \mathbf{a} . The Bruhat order on W_a extend in a natural way to \tilde{W} .

Let G_{sc} be the simply connected cover of the derived group of G . Denote by T_{sc} the maximal torus of G_{sc} given by the choice of T . Then we have a natural injective map $X_*(T_{sc})_{\Gamma} \rightarrow X_*(T)_{\Gamma}$. We fix a special vertex of \mathbf{a} and represent \tilde{W} and W_a as

$$\begin{aligned} \tilde{W} &= X_*(T)_{\Gamma} \rtimes W_0 = \{t^{\lambda}w; \lambda \in X_*(T)_{\Gamma}, w \in W_0\}, \\ W_a &= X_*(T_{sc})_{\Gamma} \rtimes W_0 = \{t^{\lambda}w; \lambda \in X_*(T_{sc})_{\Gamma}, w \in W_0\}. \end{aligned}$$

1.3. For any $w \in \tilde{W}$, we choose a representative in $N(L)$ and also write it as w . By [He2, §3], any σ -conjugacy class of $G(L)$ contains an element in \tilde{W} . The restriction of Kottwitz map and Newton map on $\tilde{W} \subset G(L)$ can be described explicitly as follows.

The map $N(L) \rightarrow G(L)$ induces a map $\tilde{W} \rightarrow B(G)$. Here $\kappa_G(w)$ is the image of w under the projection $\tilde{W} \rightarrow \Omega \cong \pi_1(G)_\Gamma \rightarrow \pi_1(G)_{\Gamma_F}$.

For any $w \in \tilde{W}$, we consider the element $w\sigma \in \tilde{W} \rtimes \langle \sigma \rangle$. There exists $n \in \mathbb{N}$ such that $(w\sigma)^n = t^\lambda$ for some $\lambda \in X_*(T)_\Gamma$. Let $\nu_{w,\sigma} = \lambda/n$ and $\bar{\nu}_{w,\sigma}$ the unique dominant element in the W_0 -orbit of $\nu_{w,\sigma}$. It is known that $\nu_{w,\sigma}$ is independent of the choice of n and is Γ -invariant. Moreover, $\bar{\nu}_{w,\sigma}$ is the Newton point of w when regarding w as an element in $G(L)$.

1.4. Let \mathbb{S} and $\tilde{\mathbb{S}}$ be the set of simple reflections of W_0 and W_a respectively. Then $\sigma(\tilde{\mathbb{S}}) = \tilde{\mathbb{S}}$. In general \mathbb{S} is not σ -stable since the special vertex of \mathbf{a} may not be σ -invariant. However, we may write σ as $\sigma = \tau \circ \sigma_0$, where σ_0 is a diagram automorphism of W_0 and the induced action of τ on the adjoint group G_{ad} is inner.

Let N be the order of σ_0 . For $\mu \in X_*(T)$, we set

$$\mu^\diamond = \frac{1}{N} \sum_{i=0}^{N-1} \sigma_0^i(\mu) \in X_*(T)_\mathbb{Q}.$$

Let μ^\sharp be the image of μ in $\pi_1(G)_{\Gamma_F}$. Set

$$B(G, \mu) = \{[b] \in B(G); \kappa_G(b) = \mu^\sharp, \nu_b \leq \mu^\diamond\}.$$

The elements in $B(G, \mu)$ are called the (neutral) acceptable elements for μ .

Let $\underline{\mu}$ be the image of μ in $X_*(T)_\Gamma$. The μ -admissible set is defined as

$$\text{Adm}(\mu) = \{w \in \tilde{W}; w \leq t^{x(\underline{\mu})} \text{ for some } x \in W_0\}.$$

Now we may reformulate the main theorem 0.1 as follows.

Theorem 1.1. *We keep the notation as in §1.4. Set $B(\tilde{W}, \mu, \sigma) = \{\bar{\nu}_{w,\sigma}; w \in t^\mu W_a, \bar{\nu}_{w,\sigma} \leq \mu^\diamond\}$. Then*

- (1) *The set $B(\tilde{W}, \mu, \sigma)$ contains a unique maximal element ν .*
- (2) *There exists an element $w \in \text{Adm}(\mu)$ with $\bar{\nu}_{w,\sigma} = \nu$.*

2. THE MAXIMAL ELEMENT IN $B(G, \mu)$

2.1. Let G_{ad} be the adjoint group of G , i.e., the quotient of G by its center. Since the buildings of G and G_{ad} coincide, the choice of an alcove \mathbf{a} in the building of G determines an alcove of G_{ad} . Then the Iwahori-Weyl group \tilde{W}_{ad} of G_{ad} is $X_*(T_{ad})_\Gamma \rtimes W_0$. Let $\pi : G \rightarrow G_{ad}$ be

the projection map. Set $T_{ad} = \pi(T)$. Then π induces maps $\tilde{W} \rightarrow \tilde{W}_{ad}$ and $X_*(T)_{\mathbb{Q}}^+ \rightarrow X_*(T_{ad})_{\mathbb{Q}}^+$, which we still denote by π .

It is easy to see that $\pi(\nu_{w,\sigma}) = \nu_{\pi(w),\sigma}$ for $w \in \tilde{W}$ and π induces a bijection of posets from $B(\tilde{W}, \mu, \sigma)$ to $B(\tilde{W}_{ad}, \pi(\mu), \sigma)$. Thus Theorem 1.1 holds for $B(\tilde{W}, \mu, \sigma)$ if and only if it holds for $B(\tilde{W}_{ad}, \pi(\mu), \sigma)$.

2.2. In the rest of this section, we assume that G is adjoint. We write σ as $\sigma = \text{Ad}(\tau) \circ \sigma_0$, where $\tau \in \tilde{W}$ is a length zero element and s_0 is a diagram automorphism of W_0 . Then $\nu_{w,\sigma} = \nu_{w\tau,\sigma_0}$ for all $w \in \tilde{W}$.

Set $V = X_*(T)_{\Gamma} \otimes_{\mathbb{Z}} \mathbb{R}$. For any $i \in \mathbb{S}$, let $\omega_i^{\vee} \in V$ be the fundamental coweight and $\alpha_i^{\vee} \in V$ be the simple coroot. We denote by $\omega_i, \alpha_i \in V^*$ the fundamental weight and simple root, respectively.

We also fix $\lambda \in X_*(T)_+$ such that $\tau \in t^{\lambda}W_0$. For each σ_0 -orbit c of \mathbb{S} , we set $\omega_c = \sum_{i \in c} \omega_i$, where ω_i is the fundamental weight for i . For any $v \in X_*(T)_{\mathbb{Q}}^+$, we set $J(v) = \{s \in \mathbb{S}; s(v) = v\}$ and $I(v) = \mathbb{S} \setminus J(v)$. If $v = \sigma_0(v)$, then both $J(v)$ and $I(v)$ are σ_0 -stable.

The follow lemma is essentially contained in [Ch]. Due to its importance, we provide a proof for completeness.

Lemma 2.1. *Let $v \in X_*(T)_{\mathbb{Q}}^+$ with $\sigma_0(v) = v$. Then $v = \nu_{w,\sigma}$ for some $w \in t^{\mu}W_a$ if and only if $\langle \omega_c, \mu^{\diamond} + \lambda^{\diamond} - v \rangle \in \mathbb{Z}$ for any σ_0 -orbit c of $I(v)$.*

Proof. Since $\nu_{w,\sigma} = v$, we have $w\tau = t^{\gamma}x$ for some $\gamma \in X_*(T)_{\Gamma}$ and $x \in W_{J(v)}$. Let N_0 be the order of $W_0 \rtimes \langle \sigma_0 \rangle$. Then

$$\begin{aligned} \nu_{w,\sigma} &= \nu_{w\tau,\sigma_0} = \frac{1}{N_0} \sum_{k=0}^{N_0-1} (x\sigma_0(x) \cdots \sigma_0^{k-1}(x))\sigma_0^k(\gamma) \\ &\in \frac{1}{N_0} \sum_{k=0}^{N_0-1} \sigma_0^k(\gamma) + \sum_{j \in J(v)} \mathbb{Q}\alpha_j^{\vee} \\ &= \gamma^{\diamond} + \sum_{j \in J(v)} \mathbb{Q}\alpha_j^{\vee}. \end{aligned}$$

If $w \in t^{\mu}W_a$, then $w\tau \in t^{\mu+\lambda}W_a$ and $\mu + \lambda - \gamma \in X_*(T_{sc})_{\Gamma}$. Hence $\langle \omega_c, \mu^{\diamond} + \lambda^{\diamond} - \gamma^{\diamond} \rangle = \langle \omega_c, \mu + \lambda - \gamma \rangle \in \mathbb{Z}$.

On the other hand, let $a_c = \langle \omega_c, \mu^{\diamond} + \lambda^{\diamond} - v \rangle \in \mathbb{Z}$ for each σ_0 -orbit c of $I(v)$. We construct an element $w \in t^{\mu}W_a$ such that $\nu_{w,\sigma} = v$.

For each σ_0 -orbit of $J(v)$, we choose a representative. Let x be the product of these representatives (in some order). Then x is a σ_0 -twisted Coxeter element of $W_{J(v)}$ in the sense of [Sp, 7.3]. For each σ_0 -orbit c of $I(v)$, we choose a representative i_c . Let $\alpha_{i_c}^{\vee}$ be the corresponding simple coroot in $X_*(T_{sc})_{\Gamma}$. Set $\beta = \mu + \lambda - \sum_c a_c \alpha_{i_c}^{\vee}$ and $w = t^{\beta}x\tau^{-1} \in t^{\mu}W_a$.

Write $\beta = h + r$ with $r \in \sum_{j \in J(v)} \mathbb{Q}\alpha_j^\vee$ and $h \in \sum_{i \in I(v)} \mathbb{Q}\omega_j^\vee$. Then

$$\begin{aligned} \nu_{w,\sigma} &= \nu_{w\tau,\sigma_0} = \frac{1}{N_0} \sum_{i=0}^{N_0-1} (x\sigma_0)^i(\beta) \\ &= h^\diamond + \frac{1}{N_0} \sum_{k=0}^{N_0-1} (x\sigma_0)^k(r) \\ &= h^\diamond = \mu^\diamond + \lambda^\diamond - \sum_c a_c(\alpha_{i_c}^\vee)^\diamond - r^\diamond, \end{aligned}$$

where the fourth equality is due to the fact that x is σ_0 -elliptic in $W_{J(v)}$.

Hence for any σ_0 -orbit c of $I(v)$ and any $j \in J(v)$, we have

$$\langle \omega_c, \mu^\diamond + \lambda^\diamond - \nu_{w,\sigma} \rangle = \langle \omega_c, \sum_{c'} a_{c'}(\alpha_{i_{c'}}^\vee)^\diamond \rangle = a_c = \langle \omega_c, \mu^\diamond + \lambda^\diamond - v \rangle$$

and

$$\langle \alpha_j, \mu^\diamond + \lambda^\diamond - \nu_{w,\sigma} \rangle = \langle \alpha_j, \mu^\diamond + \lambda^\diamond \rangle = \langle \alpha_j, \mu^\diamond + \lambda^\diamond - v \rangle,$$

which means $\nu_{w,\sigma} = v$ as desired. \square

Corollary 2.2. $\mu^\diamond \in B(G, \mu)$ if and only if $\langle \omega_c, \lambda^\diamond \rangle \in \mathbb{Z}$ for any σ_0 -orbit c of $I(\mu^\diamond)$. In this case, μ^\diamond is a priori the maximal Newton polygon of $B(G, \mu)$.

2.3. We follow [Ch, §6]. For any σ_0 -stable subset B of $X_*(T)_{\mathbb{Q}}^+$, we define

$$C_{\geq B} = \{v \in X_*(T)_{\mathbb{Q}}^+; \sigma_0(v) = v \text{ and } v \geq b, \forall b \in B\}.$$

We say B is *reduced* if $C_{\geq B'} \subsetneq C_{\geq B}$ for any σ_0 -stable proper subset $B' \subsetneq B$.

For any $i \in \mathbb{S}$, let

$$pr_{(i)} : V = \mathbb{R}\omega_i^\vee \oplus \sum_{j \neq i} \mathbb{R}\alpha_j^\vee \rightarrow \mathbb{R}\omega_i^\vee$$

be the projection map.

Now we prove part (1) of Theorem 1.1.

2.4. **Proof of Theorem 1.1 (1).** For any σ_0 -orbit c of \mathbb{S} and $i \in c$, we define $e_i \in \mathbb{Q}\omega_i^\vee$ by

$$\langle \omega_i, e_i \rangle = \frac{1}{\#c} \max(\{t \in \langle \omega_c, \mu^\diamond + \lambda^\diamond \rangle + \mathbb{Z}; t \leq \langle \omega_c, \mu^\diamond \rangle\} \cup \{0\}).$$

Let $E_0 = \{e_i; i \in \mathbb{S}\}$ and $E \subset E_0$ be a σ_0 -stable subset which is reduced and satisfies $C_{\geq E} = C_{\geq E_0}$. Let $I(E) = \{i \in \mathbb{S}; e_i \in E\}$. By [Ch, Theorem 6.5], there exists an element $\nu \in C_{\geq E}$ defined by

$\mu^\diamond + \lambda^\diamond - \nu \in X_*(T_{sc})_\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, $I(\nu) = I(E)$ and $\langle \omega_j, \nu \rangle = \langle \omega_j, e_j \rangle$ for $j \in I(E)$, which satisfies $C_{\geq \nu} = C_{\geq E} = C_{\geq E_0}$. Since $\mu^\diamond \in C_{\geq E}$, we have $\nu \leq \mu^\diamond$. By Lemma 2.1, $\nu \in B(\tilde{W}, \mu, \sigma)$.

Let $\nu' \in B(\tilde{W}, \mu, \sigma)$. Set $E(\nu') = \{pr_{(j)}(\nu'); j \in I(\nu')\}$. By Lemma 2.1 and the inequality $\nu' \leq \mu^\diamond$, we have, for any σ_0 -orbit c of $I(\nu')$ and $j \in c$, that

$$\sharp c \cdot \langle \omega_j, pr_{(j)}(\nu') \rangle = \sharp c \cdot \langle \omega_j, \nu' \rangle = \langle \omega_c, \nu' \rangle \in \langle \omega_c, \mu^\diamond + \lambda^\diamond \rangle + \mathbb{Z}$$

and

$$\sharp c \cdot \langle \omega_j, pr_{(j)}(\nu') \rangle \leq \sharp c \cdot \langle \omega_j, \mu^\diamond \rangle = \langle \omega_c, \mu^\diamond \rangle.$$

So $\langle \omega_j, pr_{(j)}(\nu') \rangle \leq \langle \omega_j, e_j \rangle$, that is, $pr_{(j)}(\nu') \leq e_j \leq \nu$ for $j \in I(\nu')$. By [Ch, Lemma 6.2 (i)], we deduce that $\nu' \leq \nu$. Therefore ν is the unique maximal element of $B(\tilde{W}, \mu, \sigma)$.

3. REDUCTION TO THE IRREDUCIBLE CASE

Lemma 3.1. *Let $\tau \in \Omega$. Then Theorem 1.1 holds for (\tilde{W}, μ, σ) if and only if it holds for $(\tilde{W}, \mu, \tau\sigma\tau^{-1})$.*

Proof. For any $v \in V$, $\bar{\tau}(v) = \bar{v}$. Thus $\tau \text{Adm}(\mu)\tau^{-1} = \text{Adm}(\mu)$ and $\bar{\nu}_{w, \tau\sigma\tau^{-1}} = \tau(\bar{\nu}_{\tau^{-1}w\tau, \sigma}) = \bar{\nu}_{\tau^{-1}w\tau, \sigma}$. Therefore $B(\tilde{W}, \mu, \tau\sigma\tau^{-1}) = B(\tilde{W}, \mu, \sigma)$. Since conjugation by τ preserves the Bruhat order, Theorem 1.1 (2) holds for (\tilde{W}, μ, σ) if and only if it holds for $(\tilde{W}, \mu, \tau\sigma\tau^{-1})$. \square

3.1. In the rest of this section, we assume that G is adjoint and σ acts transitively on the set of connected components of the affine Dynkin diagram of W_a . In other words, $\tilde{W} = \tilde{W}_1 \times \cdots \times \tilde{W}_m$, where $\tilde{W}_1 \cong \cdots \cong \tilde{W}_m$ are extended affine Weyl groups of adjoint type with connected affine Dynkin diagram and $\sigma(\tilde{W}_1) = \tilde{W}_2, \dots, \sigma(\tilde{W}_m) = \tilde{W}_1$. After conjugating a suitable element in Ω , we may assume that $\sigma = \text{Ad}(\tau) \circ \sigma_0$ with $\tau \in \tilde{W}_m$. We may write μ as $\mu = (\mu_1, \dots, \mu_m)$, where μ_i is a dominant coweight for \tilde{W}_i . Set $\gamma = \sum_{i=1}^m \sigma_0^{m-i}(\mu_i)$. Then the natural projection induces a bijection from $B(\tilde{W}, \mu, \sigma)$ to $B(\tilde{W}_m, \gamma, \sigma^m)$.

Lemma 3.2. *We keep the notations in §3.1. If Theorem 1.1 (2) holds for $(\tilde{W}_m, \underline{\mu}, \sigma^m)$, then it holds for (\tilde{W}, μ, σ) .*

Proof. Let ν be the maximal element in $B(\tilde{W}_m, \gamma, \sigma^m)$. By assumption, there exists $w \in \text{Adm}(\gamma)$ such that $\bar{\nu}_{w, \sigma^m} = \nu$. By definition, there exists x in the finite Weyl group associated to \tilde{W}_m such that $w \leq t^x(\underline{\gamma})$. Since $\ell(t^x(\underline{\gamma})) = \sum_{i=1}^m \ell(t^{x(\sigma_0^{m-i}(\underline{\mu}_i))})$, there exists $w_i \in \tilde{W}_m$ for each i such that $w = w_1 \cdots w_m$ and $w_i \leq t^{x(\sigma_0^{m-i}(\underline{\mu}_i))}$ for all i . Hence

$\sigma_0^{i-m}(w_i) \leq t^{x(\mu_i)}$. Set $y = (\sigma_0^{1-m}(w_i), \dots, w_m) \in \tilde{W}$. Then $y \in \text{Adm}(\mu)$ and $\nu_{y,\sigma} = (\sigma\nu_{w,\sigma^m}, \dots, \nu_{w,\sigma^m})$. Hence $\bar{\nu}_{y,\sigma} = (\sigma_0\nu, \dots, \nu)$ is the maximal element in $B(\tilde{W}, \mu, \sigma)$. \square

4. REDUCTION TO THE SUPERBASIC CASE

4.1. Let $\epsilon \in \tilde{W} \rtimes \langle \sigma \rangle$. We say that ϵ is *superbasic* if $\ell(\epsilon) = 0$ and each ϵ -orbit on \tilde{S} is a union of the connected components of the affine Dynkin diagram of \tilde{W} . By [HN1, 3.5], ϵ is superbasic if and only if $W_a = W_1^{m_1} \times \dots \times W_l^{m_l}$, where W_i is an affine Weyl group of type \tilde{A}_{n_i-1} and ϵ gives an order $n_i m_i$ permutation on the set of simple reflections of $W_i^{m_i}$.

4.2. The main purpose of this section is to reduce to the case where σ is superbasic. We keep the assumption in §2.2.

For any $J \subset \mathbb{S}$, let W_J be the subgroup of W_0 generated by s_j for $j \in J$ and ${}^J W_0$ be the set of minimal coset representatives in $W_J \backslash W_0$. Let $\tilde{W}_J = X_*(T)_\Gamma \rtimes W_J$.

We regard σ as an element in $\tilde{W} \rtimes \langle \sigma \rangle$. We will construct a superbasic element in $\tilde{W}_J \rtimes \langle \sigma_0 \rangle$ for a suitable subset $J \subset \mathbb{S}$ with $\sigma_0(J) = J$. We follow the approach in [HN2, §5].

Let V^σ be the fixed point set of σ . Since σ is an affine transformation on V of finite order, V^σ is a nonempty affine subspace. Set $V' = \{v - e; v \in V^\sigma\}$, where e is an arbitrary point of V^σ . Then V' is the (linear) subspace of V parallel to V^σ . We choose a generic point v_0 of V' , i.e., for any root α , $\langle \alpha, v \rangle = 0$ implies that $\langle \alpha, v' \rangle = 0$ for all $v' \in V'$. Let \bar{v}_0 be the unique dominant element of the W_0 -orbit of v_0 . We set $I = I(\bar{v}_0)$ and $J = J(\bar{v}_0)$. Let $z \in {}^J W_0$ be the unique element with $\bar{v}_0 = z(v_0)$. Set $\sigma^J = z\sigma z^{-1}$.

Lemma 4.1. *The element σ^J is a superbasic element in $\tilde{W}_J \rtimes \langle \sigma_0 \rangle$.*

Proof. Since $\sigma(0) = \lambda$, $\sigma(v_0) = v_0 + \lambda$ and $\sigma^J(\bar{v}_0) = \bar{v}_0 + z(\lambda)$. Write σ^J as $\sigma^J = t^{z(\lambda)} u \sigma_0$ for some $u \in W_0$. Then $u \sigma_0(\bar{v}_0) = \bar{v}_0$. Therefore $\sigma(\bar{v}_0) = u^{-1} \bar{v}_0$ is the unique dominant element in the W_0 -orbit of v_0 . Hence $\bar{v}_0 = \sigma(\bar{v}_0) = u^{-1} \bar{v}_0$. Therefore $u \in W_J$ and $\sigma_0(J) = J$.

Let ℓ_J be the length function on $\tilde{W}_J \rtimes \langle \sigma_0 \rangle$. By [HN2, Proposition 3.2], $\ell_J(\sigma^J) = 0$. Since v_0 is generic, by [HN2, §5.5], $V^{u\sigma_0} \subset V^{W_J}$. Therefore there is no nonempty subset of J that is stable under σ^J . Hence each orbit of σ^J on the set of simple reflections of \tilde{W}_J is a union of connected components of the affine Dynkin diagram of \tilde{W}_J . Hence σ^J is superbasic. \square

Lemma 4.2. *We keep the notations in §4.2. Then*

$$(1) z(\lambda)^\diamond \in \sum_{j \in J} \mathbb{Q}\alpha_j^\vee.$$

(2) *Let c be a σ_0 -orbit of \mathbb{S} . Then $\langle \omega_c, \lambda^\diamond \rangle \in \mathbb{Z}$ if and only if $c \subset I$.*

Proof. Assume $z(\lambda) \in r + h$ with $r \in \sum_{j \in J} \mathbb{Q}\alpha_j^\vee$ and $h \in \sum_{i \in I} \mathbb{Q}\omega_i^\vee$. Since σ is of length zero, we have $\nu_{tz(\lambda)u, \sigma_0} = 0$, which implies $h^\diamond = 0$. Hence $z(\lambda)^\diamond \in \sum_{j \in J} \mathbb{Q}\alpha_j^\vee$ and (1) is proved.

Write $\lambda = z(\lambda) + \theta$ for some $\theta \in X_*(T_{sc})_\Gamma$. We have

$$\langle \omega_c, \lambda^\diamond \rangle = \langle \omega_c, z(\lambda)^\diamond \rangle + \langle \omega_c, \theta \rangle = \langle \omega_c, r^\diamond \rangle + \langle \omega_c, \theta \rangle \equiv \langle \omega_c, r \rangle \pmod{\mathbb{Z}}.$$

Hence $\langle \omega_c, \lambda^\diamond \rangle \in \mathbb{Z}$ if $c \subset I$. On the other hand, since σ^J is a superbasic element of \tilde{W}_J , \tilde{W}_J has only type A factors. One may check directly that $\langle \omega_c, r \rangle = \langle \omega_c^J, r \rangle \notin \mathbb{Z}$ for any σ_0 -orbit c of I and (2) is proved. \square

Proposition 4.3. *The maximal Newton point of $B(\tilde{W}, \mu, \sigma)$ is contained in the natural inclusion $B(\tilde{W}_J, \mu, \sigma^J) \hookrightarrow B(\tilde{W}, \mu, \sigma)$.*

Proof. We denote by $\omega_j^J \in \sum_{j \in J} \mathbb{Q}\alpha_j^\vee$ the corresponding fundamental coweight of Φ_J and set $\omega_c^J = \sum_{j \in c} \omega_j^J$ for any σ_0 -orbit c of J . Let ν be the maximal Newton point of $B(\tilde{W}, \mu, \sigma)$. By the proof of Theorem 1.1 (1), for each σ_0 -orbit c of \mathbb{S} ,

$$\langle \omega_c, \mu^\diamond \rangle \geq \langle \omega_c, \nu \rangle \geq \langle \omega_c, \mu^\diamond + \lambda^\diamond \rangle - \lceil \langle \omega_c, \lambda^\diamond \rangle \rceil.$$

Let c be a σ_0 -orbit of I . By Lemma 4.2 (2), $\langle \omega_c, \lambda^\diamond \rangle \in \mathbb{Z}$. Hence $\langle \omega_c, \mu^\diamond \rangle = \langle \omega_c, \nu \rangle$ and $\mu^\diamond - \nu \in \sum_{j \in J} \mathbb{Q}\alpha_j^\vee$.

By Lemma 4.2 (1), $z(\lambda)^\diamond \in \sum_{j \in J} \mathbb{Q}\alpha_j^\vee$. Thus $\mu^\diamond + z(\lambda)^\diamond - \nu \in \sum_{j \in J} \mathbb{Q}\alpha_j^\vee$. Now c' be a σ_0 -orbit in $I(\nu) \cap J$. Then

$$\langle \omega_{c'}^J, \mu^\diamond + z(\lambda)^\diamond - \nu \rangle = \langle \omega_{c'}, \mu^\diamond + z(\lambda)^\diamond - \nu \rangle = \langle \omega_c, \mu^\diamond + \lambda^\diamond - \nu \rangle - \langle \omega_{c'}, \theta \rangle,$$

where $\theta = \lambda - z(\lambda) \in X_*(T_{sc})_\Gamma$. by Lemma 2.1 $\langle \omega_{c'}, \mu^\diamond + \lambda^\diamond - \nu \rangle \in \mathbb{Z}$.

Hence $\langle \omega_{c'}^J, \mu^\diamond + z(\lambda)^\diamond - \nu \rangle \in \mathbb{Z}$. Again by Lemma 2.1, $\nu \in B(\tilde{W}_J, \mu, \sigma^J)$. \square

Lemma 4.4. *Let $J \subset \mathbb{S}$ and $x \in {}^J W_0$. If $w, w' \in \tilde{W}_J$ with $w \leq_J w'$ for the Bruhat order of \tilde{W}_J , then $z^{-1}wz \leq z^{-1}w'z$ for the Bruhat order of \tilde{W} .*

Proof. It suffices to consider the case where $w' = ws_\alpha$ for some positive affine root α of \tilde{W}_J . Since $w' \geq w$, $w(\alpha)$ is again a positive affine root of \tilde{W}_J . Since x^{-1} sends positive affine roots of \tilde{W}_J to positive affine roots of \tilde{W} , $z^{-1}w'z = z^{-1}ws_{z^{-1}(\alpha)}$ and $z^{-1}w(\alpha)$ is a positive affine root of \tilde{W} . Hence $z^{-1}w'z \geq z^{-1}wz$. \square

Corollary 4.5. *If Theorem 1.1 (2) holds for $B(\tilde{W}_J, \mu, \sigma^J)$, then it holds for $B(\tilde{W}, \mu, \sigma)$.*

Proof. Let ν be the maximal Newton point of $B(\tilde{W}, \mu, \sigma)$, which is also the maximal Newton point of $B(\tilde{W}_J, \mu, \sigma^J)$ by Proposition 4.3. By assumption, there exist $w_1 \in t^\mu(W_a \cap \tilde{W}_J)$ and $x_1 \in W_J$ such that $\bar{\nu}_{w_1, \sigma^J}^J = \nu$ and $w_1 \leq_J t^{x_1(\mu)}$. Here \leq_J is the Bruhat order on \tilde{W}_J defined with respect to J . Let $w = z^{-1}w_1z$ and $x = z^{-1}x_1$. Then we have $\bar{\nu}_{w, \sigma} = \nu$, $w \in t^\mu W_a$ and $w \leq t^{x(\mu)}$ as desired. \square

5. THE SUPERBASIC CASE

5.1. In this section, we consider the extended affine Weyl group \tilde{W} of $G = GL_n$. Then $\tilde{W} \cong \mathbb{Z}^n \rtimes \mathfrak{S}_n$, where \mathfrak{S}_n is the permutation group of $\{1, 2, \dots, n\}$ which acts on $\mathbb{Z}^n \cong \bigoplus_{i=1}^n \mathbb{Z}e_i^\vee$ by $w(e_i^\vee) = e_{w(i)}^\vee$ for $w \in \mathfrak{S}_n$. Let $\{e_i\}_{i=1, \dots, n}$ be the dual basis. Set $d = \sum_{i=1}^n e_i$ and $d^\vee = \sum_{i=1}^n e_i^\vee$. The simple roots and fundamental weights are given by $\alpha_i = e_i - e_{i+1}$ and $\omega_i = -\frac{i}{n}d + \sum_{j=1}^i e_j$ respectively for $i \in [1, n-1]$. Set $\varpi_{m,n} = \sum_{j=1}^i e_j^\vee$.

For any positive integer $m < n$, let $\sigma_{m,n} = t^{\varpi_{m,n}} u_{m,n} \in t^{\varpi_{m,n}} W_0$ be the unique length zero element with $u_m \in W_0$. Then any superbasic element in \tilde{W} is of the form $t^{\varpi_{m,n}} \sigma_{m,n}$ for some m coprime to n .

The main purpose of this section is to prove the following result.

Proposition 5.1. *Let $m < n$ be a positive integer coprime to n . Let $\mu \in \mathbb{Z}^n$ be a dominant coweight of $G = GL_n$. Then there exists $\tilde{w} \in \tilde{W}$ and $x \in W_0$ such that $\tilde{w} < t^{x(\mu)}$ and $\bar{\nu}_{\tilde{w}, \sigma_{m,n}} - \frac{m}{n}d^\vee$ equals the unique maximal Newton point ν of $B(\tilde{W}, \mu, t^{-\frac{m}{n}d^\vee} \sigma_{m,n})$.*

The proof will be given in §5.6.

5.2. We first show that Proposition 5.1 implies Theorem 1.1 (ii).

By §2.1, we may assume that \tilde{W} is the Iwahori-Weyl group of an adjoint p -adic group G . Then by Lemma 3.2 and Lemma 4.5, it suffices to prove the case where $G = PGL_n$ and σ is superbasic, which follows from Proposition 5.1 and §2.1.

5.3. Now we give an algorithm to construct the maximal element in $B(\tilde{W}, \mu, \sigma_{m,n})$.

We recall the definition of \mathbf{a} -sequence and $\chi_{m,n}$ in [He1, §3 & §5].

Let $r \in \mathbb{N}$ and $\chi \in \mathbb{Z}^r$. For each $j \in [1, r]$ we define $\mathbf{a}_\chi^j : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}$ by $\mathbf{a}_\chi^j(k) = \chi(j-k)$. Here we identify l with $l+r$ for $l \in \mathbb{Z}$. We say $i \geq_\chi j$ if $\mathbf{a}_\chi^i \geq \mathbf{a}_\chi^j$ in the sense of lexicographic order. If \geq_χ is a linear order, we define $\epsilon_\chi \in \mathfrak{S}_r$ such that $\epsilon_\chi(i) < \epsilon_\chi(j)$ if and only if $i >_\chi j$.

Define $\chi_{m,n} \in \mathbb{Z}^n$ by $\chi_{m,n}(i) = \lfloor im \rfloor - \lfloor (i-1)m \rfloor$ for $i \in [1, n]$. Set $\epsilon_{m,n} = \epsilon_{\chi_{m,n}} \in \mathfrak{S}_n$. Since m and n are co-prime, it is well defined. Note that $\epsilon_{m,n}(\chi_{m,n}) = \varpi_{m,n}$.

5.4. Let $\mathcal{S} = \cup_{1 \leq i \leq j \leq n} \mathbb{Z}^{[i,j]}$, whose elements are called *segments*. Let $\eta \in \mathcal{S}$ be a segment. Assume $\eta \in \mathbb{Z}^{[i,j]}$. We call $h(\eta) = i$ and $t(\eta) = j$ the *head* and the *tail* of η respectively. We call the nonnegative integer $j - i + 1$ the *size* of η . We set

$$|\eta| = \sum_{k=h(\eta)}^{t(\eta)} \eta(k), \quad \text{av}(\eta) = \frac{1}{t(\eta) - h(\eta) + 1} |\eta|.$$

Let $[i', j'] \subset [i, j]$ be a sub-interval, we call the restriction $\eta|_{[i', j']}$ of η to $[i', j]$ a *subsegment* of η and write $\eta|_i = \eta|_{[i, i]}$. Let θ be another segment such that $h(\theta) = t(\eta) + 1$. We denote by $\eta \vee \theta \in \mathbb{Z}^{[h(\eta), t(\theta)]}$ the natural union of η and θ . For $k \in \mathbb{Z}$, we denote by $\eta[k]$ the k -*shift* of η defined by $\eta[k](i) = \eta(i + k)$. We say two segments are of the same *type* if they can be identified with each other up to some shift.

For $\eta \in \mathbb{Q}^n \cong \mathbb{Q}^{[1,n]}$ we denote by $\text{Con}(\eta) \in \mathbb{Q}^2$ the convex hull of the points $(0, 0)$ and $(k, |\eta|_{[1,k]})$ for $k \in [1, n]$. We say a subsegment γ of η is *sharp* if $\text{av}(\gamma)$ is maximal/minimal among all subsegments of η with the same head/tail. If $\eta = \gamma^1 \vee \gamma^2 \vee \cdots \vee \gamma^s$ with each γ^k a sharp subsegments, then the points $(0, 0)$ and $(t(\gamma^i), |\gamma^1 \vee \cdots \vee \gamma^i|)$ in \mathbb{R}^2 for $i \in [1, s]$ lie on the boundary of $\text{Con}(\eta)$ and their convex hull is just $\text{Con}(\eta)$. We call the dominant vector

$$\text{sl}(\text{Con}(\eta)) = (\mathbf{av}(\gamma^1) \vee \cdots \vee \mathbf{av}(\gamma^s)) \in \mathbb{Q}^s$$

the *slope sequence* of $\text{Con}(\eta)$. Here for any $\gamma \in \mathcal{S}$, we define $\mathbf{av}(\gamma) \in \mathbb{Z}^{[h(\gamma), t(\gamma)]}$ by $\mathbf{av}(\gamma)(i) = \text{av}(\gamma)$ for $i \in [h(\gamma), t(\gamma)]$.

Let $\mu \in \mathbb{Z}^n$ be a dominant coweight. Define $\mu_{m,n} = \mu + \chi_{m,n}$. Then

$$\begin{aligned} \langle \omega_i, \mu_{m,n} \rangle &= \langle \omega_i, \mu \rangle - \left(\frac{mi}{n} - \left\lfloor \frac{mi}{n} \right\rfloor \right) \\ &= \langle \omega_i, \mu + \varpi_{m,n} \rangle - \lceil \langle \omega_i, \varpi_{m,n} \rangle \rceil. \end{aligned}$$

According to the proof of Theorem 1.1 (1), the slope sequence $\nu = \text{sl}(\text{Con}(\mu_{m,n}))$ is the unique maximal Newton point of $B(G, \mu, \sigma_{m,n})$.

Example 5.2. Now we provide an example.

For a sequence of (distinct) elements i_1, i_2, \dots, i_r in $[1, n]$, we denote by $\text{cyc}(i_1, i_2, \dots, i_r) \in \mathfrak{S}_n$ the cyclic permutation $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_r \mapsto i_1$, which acts trivially on the remaining elements of $[1, n]$.

Let $n = 8$, $m = 5$ and $\mu = (1, 1, 1, 0, 0, 0, 0, 0)$. Then $\chi_{m,n} = (0, 1, 0, 1, 1, 0, 1, 1) \in \mathbb{Z}^8$, $\epsilon_{m,n} = \text{cyc}(1, 6, 7, 4, 5, 2, 3, 8)$ and $u_{m,n} = \text{cyc}(6, 3, 8, 5, 2, 7, 4, 1)$.

We have the following sharp decomposition:

$$\mu_{m,n} = (1, 2, 1, 1, 1, 0, 1, 1) = (1, 2) \vee (1) \vee (1, 1) \vee (0, 1, 1).$$

Hence $\nu = (\frac{3}{2}, \frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. Moreover, one checks that

$$\begin{aligned} t^{\epsilon_{m,n}(\mu)} \sigma_{m,n} &> t^{\epsilon_{m,n}(\mu)} \sigma_{m,n} \text{cyc}(8, 3) \\ &> t^{\epsilon_{m,n}(\mu)} \sigma_{m,n} \text{cyc}(8, 3) \text{cyc}(1, 3) \\ &> t^{\epsilon_{m,n}(\mu)} \sigma_{m,n} \text{cyc}(8, 3) \text{cyc}(1, 3) \text{cyc}(1, 2). \end{aligned}$$

Set $\tilde{w} = t^{\epsilon_{m,n}(\mu)} \sigma_{m,n} \text{cyc}(8, 3) \text{cyc}(1, 3) \text{cyc}(1, 2) \sigma_{m,n}^{-1}$. Then $\tilde{w} < t^{\epsilon_{m,n}(\mu)}$ and $\bar{\nu}_{\tilde{w}, \sigma_{m,n}} = \nu$. This verifies Proposition 5.1 in this case.

5.5. Similar to [He1, §5], we use the Euclidean algorithm to give a recursive construction of $\chi_{m,n}$, which plays a crucial role in the proof of Proposition 5.1.

Let $D = \{(i, j) \in \mathbb{Z}_{>0}^2; i < j \text{ are co-prime}\}$. We define $f : D \rightarrow D \sqcup \{(1, 1)\}$ by

$$f(m, n) = \begin{cases} (m(\lfloor \frac{n}{m} \rfloor + 1) - n, m), & \text{if } \frac{n}{m} \geq 2; \\ (n - (n - m)\lfloor \frac{n}{n-m} \rfloor, n - m), & \text{otherwise.} \end{cases}$$

Define two types of segments $1_{m,n}$ and $0_{m,n}$ in \mathcal{S} by

$$(1_{m,n}, 0_{m,n}) = \begin{cases} ((0(\lfloor \frac{n}{m} \rfloor - 1), 1), (0(\lfloor \frac{n}{m} \rfloor), 1)), & \text{if } \frac{n}{m} \geq 2; \\ ((0, 1(\lfloor \frac{n}{n-m} \rfloor)), (0, 1(\lfloor \frac{n}{n-m} \rfloor - 1))), & \text{otherwise.} \end{cases}$$

For $\eta \in \mathcal{S}$ and $k \in [\text{h}(\eta), \text{t}(\eta)]$, set

$$\eta(k)_{m,n} = \begin{cases} 1_{m,n}, & \text{if } \eta(k) = 1; \\ 0_{m,n}, & \text{otherwise.} \end{cases}$$

For $k \in [\text{h}(\eta), \text{t}(\eta)]$, let $\eta_{m,n,k}$ be the shift of $\eta(k)_{m,n}$ whose head is determined recursively as follows:

$$\text{h}(\eta_{m,n,k}) = \begin{cases} \text{h}(\eta), & \text{if } k = \text{h}(\eta); \\ \text{t}(\eta_{m,n,k-1}) + 1, & \text{if } k > \text{h}(\eta). \end{cases}$$

Now we define $\phi_{m,n} : \mathcal{S} \rightarrow \mathcal{S}$ by $\phi_{m,n}(\eta) = \eta_{m,n,\text{h}(\eta)} \vee \cdots \vee \eta_{m,n,\text{t}(\eta)}$ for $\eta \in \mathcal{S}$.

If $f^{h-1}(m, n)$ is defined, we set $\phi_{m,n,h} = \phi_{f^{h-1}(m,n)} \circ \cdots \circ \phi_{m,n}$. Using the Euclidean algorithm, one checks that

$$\phi_{m,n,h}(\chi_{f^h(m,n)}) = \chi_{m,n}.$$

We say a subsegment γ of $\chi_{m,n}$ is of *level* h if it is the image of some subsegment γ^h of $\chi_{f^h(m,n)}$ under the map $\phi_{m,n,h}$. When $h = 1$ and γ^h is of size one, we say γ is an *elementary* subsegment of $\chi_{m,n}$.

Let β^1 and γ^1 be two segments of $\chi^1 = \chi_{f(m,n)}$ and let γ be a level one subsegment of $\chi = \chi_{m,n}$. Using the Euclidean algorithm, we have the following basic facts:

- (a) $\text{av}(\beta^1) \geq \text{av}(\gamma^1)$ if and only if $\text{av}(\phi_{m,n}(\beta^1)) \geq \text{av}(\phi_{m,n}(\gamma^1))$.
- (b) Each sharp subsegment of γ with the same head is of level one.
- (c) If moreover γ is an elementary subsegment of χ , then $\underline{\mathbf{a}}_\chi^j < \underline{\mathbf{a}}_\chi^{h(\gamma)-1}$ and $\underline{\mathbf{a}}_\chi^j < \underline{\mathbf{a}}_\chi^{t(\gamma)}$ for $j \in [h(\gamma), t(\gamma) - 1]$.
- (d) $\underline{\mathbf{a}}_{\chi^1}^i < \underline{\mathbf{a}}_{\chi^1}^j$ if and only if $\underline{\mathbf{a}}_\chi^{t(\phi_{m,n}(\chi^1|_{[i]}))} < \underline{\mathbf{a}}_\chi^{t(\phi_{m,n}(\chi^1|_{[j]}))}$.
- (e) $\epsilon_{m,n}(n) = 1$.

5.6. Proof of Proposition 5.1. For $\eta \in \mathcal{S}$ we set $x_\eta = \text{cyc}(h(\eta), h(\eta) + 1, \dots, t(\eta)) \in \mathfrak{S} = \cup_{i=1}^\infty \mathfrak{S}_i$. Similarly, for a sequence $\mathbf{c} = (c^1, \dots, c^s)$ of segments, we set $x_{\mathbf{c}} = x_{c^1, \dots, c^s} = x_{c^1} \cdots x_{c^s}$. If $\eta = c^1 \vee \cdots \vee c^s$, we say \mathbf{c} is *decomposition* of η . Now we are ready to prove Proposition 5.1.

Write $\chi = \chi_{m,n}$, $\theta = \mu_{m,n}$ and $\epsilon = \epsilon_{m,n}$. For $h \in \mathbb{Z}_{>0}$ we set $\phi_h = \phi_{m,n,h}$ and $\chi^h = \chi_{f^h(m,n)}$. By §5.4, we have $\nu = \text{sl}(\text{Con}(\theta))$. The proof will proceed as follows. First we construct a suitable sharp decomposition \mathbf{c} of θ . One checks directly $\nu_{w_{\mathbf{c}}, \text{id}} = \text{sl}(\text{Con}(\theta)) = \nu$, where $w_{\mathbf{c}} = t^\theta x_{\mathbf{c}} \in \tilde{W}$. Then we show that

$$\epsilon w_{\mathbf{c}} \epsilon^{-1} < t^{\epsilon(\mu)} \sigma_{m,n} = \epsilon t^\theta x_\theta \epsilon^{-1}.$$

Set $\tilde{w} = \epsilon w_{\mathbf{c}} \epsilon^{-1} \sigma_{m,n}^{-1}$. Then $\tilde{w} < t^{\epsilon(\mu)}$ and $\nu_{\tilde{w}, \sigma_{m,n}} = \nu_{\epsilon w_{\mathbf{c}} \epsilon^{-1}, \text{id}} = \epsilon(\nu)$. This completes our proof.

Assume $I(\mu) = \{j \in [1, n-1]; \langle \alpha_j, \mu \rangle \neq 0\} = \{b_1, b_2, \dots, b_{r-1}\}$ with $b_1 < b_2 < \cdots < b_{r-1}$. We set $b_0 = 0$ and $b_r = n$. Set $\theta^i = \theta|_{[b_{i-1}+1, b_i]}$ for $i \in [1, r]$. Then $\theta = \theta^1 \vee \cdots \vee \theta^r$. Suppose we have a sharp decomposition \mathbf{c}_i of θ^i for each $i \in [1, r]$. Since $\chi \in \{0, 1\}^{[1, n]}$ and $\theta = \mu + \chi$, for any subsegment η^i (resp. η^j) of θ^i (resp. θ^j) we have $\text{av}(\eta^i) \geq \text{av}(\eta^j)$ if $i < j$. Therefore the natural union $\mathbf{c} = \mathbf{c}_1 \vee \cdots \vee \mathbf{c}_r$ forms a sharp decomposition of θ .

Let $1 \leq i \leq r$. We will construct inductively the subsegments $\zeta_i^j, \gamma_i^j, \xi_i^j$ for $j \in [1, l_i]$ (some of them may be empty) such that

- (a) $\gamma_i^0 = \theta^i$ and $\gamma_i^{j-1} = \zeta_i^j \vee \gamma_i^j \vee \xi_i^j$ for $j \in [1, l_i]$;
- (b) ζ_i^j and ξ_i^j are sharp subsegments of γ_i^{j-1} ; any sharp subsegment of γ_i^j is also a sharp subsegment of γ_i^{j-1} ; $\gamma_i^{l_i}$ is a sharp subsegment of itself (*self-sharp*);

(c) For any j , $\epsilon z_{i,j-1} \epsilon^{-1} > \epsilon z_{i,j} \epsilon^{-1}$.

Here

$$\begin{aligned} z_{i,j} &= t^\theta y_{i-1} x_i^j v_{i,j}; \\ y_i &= x_{\mathbf{c}_1} \cdots x_{\mathbf{c}_{i-1}}; \\ x_i^j &= x_{\zeta_i^1, \dots, \zeta_i^j, \xi_i^j, \dots, \xi_i^1}; \\ v_{i,j} &= x_{\gamma_i^j} x_{\theta^{i+1} \vee \dots \vee \theta^r} \text{cyc}(t(\gamma_i^j), n)) \\ &= \text{cyc}(h(\gamma_i^j), \dots, t(\gamma_i^j), b_i + 1, \dots, n). \end{aligned}$$

Once we have (a), (b) and (c) for all i and j , then

$$\mathbf{c}_i = (\zeta_i^1, \zeta_i^2, \dots, \zeta_i^{l_i}, \gamma_i^{l_i}, \xi_i^{l_i}, \dots, \xi_i^2, \xi_i^1)$$

forms a sharp decomposition of θ^i , and

$$\epsilon t^\theta x_\theta \epsilon^{-1} = \epsilon z_{1,0} \epsilon^{-1} > \cdots > \epsilon z_{1,l_1+1} \epsilon^{-1} = \epsilon z_{2,0} \epsilon^{-1} > \cdots > \epsilon z_{r,l_r+1} \epsilon^{-1} = \epsilon w_{\mathbf{c}} \epsilon^{-1}$$

as desired.

The construction is as follows. Suppose for $1 \leq k < i$ and $0 \leq l \leq j$, \mathbf{c}_k , z_i^l , ξ_i^l , γ_i^l are already constructed, and moreover $\epsilon z_{i,j-1} \epsilon^{-1} > \epsilon z_{i,j} \epsilon^{-1}$. We construct ζ_i^{j+1} , γ_i^{j+1} , ξ_i^{j+1} and show that $\epsilon z_{i,j} \epsilon^{-1} > \epsilon z_{i,j+1} \epsilon^{-1}$.

If γ_i^j is empty, there is nothing to do. Otherwise, we assume γ_i^j is of level h but not of level $h+1$. Then $\gamma_i^j = \phi_h(\iota)$ for some subsegment ι of χ^h .

Case (I): ι is not a subsegment of any elementary subsegment of χ^h . Then there exist unique subsegments ζ , γ and ξ of χ^h such that γ is of level one, ζ (resp. ξ) is a proper subsegment of some elementary segment of χ^h with the same tail (resp. head), and $\iota = \zeta \vee \gamma \vee \xi$. Without loss of generality, we may assume that none of γ , ζ and ξ is empty.

Define $\zeta_i^{j+1} = \phi_h(\zeta)$, $\gamma_i^{j+1} = \phi_h(\gamma)$ and $\xi_i^{j+1} = \phi_h(\xi)$. Note that $\text{av}(\chi^h|_{[\mathbf{h}(\zeta), \mathbf{t}(\zeta)]})$ (resp. $\text{av}(\chi^h|_{[\mathbf{h}(\xi), \mathbf{t}(\xi)]})$) is maximal (resp. minimal) among all subsegments of χ^h with the same head (resp. tail). Therefore, (b) follows from §5.5 (a) & (b). For (c), it suffices to show that

$$(d) \quad \epsilon z_{i,j+1} \epsilon^{-1} < \epsilon z_{i,j} \text{cyc}(n, t(\zeta_i^{j+1})) \epsilon^{-1};$$

$$(e) \quad \epsilon z_{i,j} \text{cyc}(n, t(\zeta_i^{j+1})) \epsilon^{-1} < \epsilon z_{i,j} \epsilon^{-1}.$$

Note that $\epsilon z_{i,j+1} \epsilon^{-1} = \epsilon z_{i,j} \text{cyc}(n, t(\zeta_i^{j+1})) \text{cyc}(h(\xi_i^{j+1}) - 1, t(\xi_i^{j+1})) \epsilon^{-1}$. By §5.5 (c), we have $\mathbf{a}_{\chi^h}^{h(\xi)-1} > \mathbf{a}_{\chi^h}^{t(\xi)}$. Hence by §5.5 (d), $\epsilon(h(\xi_i^{j+1}) - 1) < \epsilon(t(\xi_i^{j+1}))$. If $\theta(h(\xi_i^{j+1})) > \theta(b_i + 1)$, then (d) holds. If $\theta(h(\xi_i^{j+1})) =$

$\theta(b_i + 1)$, then $\chi(b_i + 1) = 1 > 0 = \chi(h(\xi_i^{j+1}))$. Hence $\epsilon(b_i + 1) < \epsilon(h(\xi_i^{j+1}))$. Then (d) still holds.

Since $\gamma \neq \emptyset$, $t(\zeta_i^{j+1}) \neq n$. By §5.5(e), $1 = \epsilon(n) < \epsilon(t(\zeta_i^{j+1}))$. If $\theta(t(\zeta_i^{j+1}) + 1) < \theta(h(\zeta_i^{j+1}))$, then (e) holds. If $\theta(t(\zeta_i^{j+1}) + 1) = \theta(h(\zeta_i^{j+1}))$, then $\chi(h(\zeta_i^{j+1})) = \chi(t(\zeta_i^{j+1}) + 1) = 0$. Since ζ is a proper subsegment of some elementary segment of χ^h and shares the same tail with it, by §5.5 (c) we have that $\underline{a}_{\chi^h}^{t(\zeta)} > \underline{a}_{\chi^h}^{h(\zeta)-1}$, and by §5.5 (d) we have that $\underline{a}_{\chi}^{t(\zeta_i^{j+1})+1} > \underline{a}_{\chi}^{h(\zeta_i^{j+1})}$. So $\epsilon(h(\zeta_i^{j+1})) > \epsilon(t(\zeta_i^{j+1}) + 1)$. Then (e) still holds.

Case (II): ι is a subsegment of some elementary subsegment of χ^h . We define $l_i = j$ and the construction of \mathbf{c}_i is finished. One checks directly that ι is self-sharp, hence so is $\gamma_i^{l_i} = \phi_h(\iota)$ by §5.5 (a) & (b). If $t(\gamma_i) = n$, the induction step is finished. Otherwise, it remains to show

$$(f) \quad \epsilon z_{i,l_i} \epsilon^{-1} > \epsilon z_{i,l_i} \text{cyc}(t(\gamma_i^{l_i}), n) \epsilon^{-1} = \epsilon z_{i,l_i+1} \epsilon^{-1}.$$

Note that $1 = \epsilon(n) < \epsilon(t(\gamma_i))$. If $\theta(h(\gamma_i^{l_i})) > \theta(b_i + 1)$, (f) holds. Otherwise, we have $\theta(h(\gamma_i^{l_i})) = \theta(b_i + 1)$, $\chi(h(\gamma_i^{l_i})) = 0$ and $\chi(b_i + 1) = 1$ since $b_i \in I(\mu)$. Hence $\epsilon(h(\gamma_i^{l_i})) > \epsilon(b_i + 1)$ and (f) still holds.

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